The Bohr-van Leeuwen theorem and the thermal Casimir effect for conductors

Giuseppe Bimonte*

Dipartimento di Scienze Fisiche Università di Napoli Federico II Complesso Universitario MSA, Via Cintia I-80126 Napoli Italy and INFN Sezione di Napoli, ITALY

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The problem of estimating the thermal corrections to Casimir and Casimir-Polder interactions in systems involving conducting plates has attracted considerable attention in the recent literature on dispersion forces. Alternative theoretical models, based on distinct low-frequency extrapolations of the plates reflection coefficient for transverse electric (TE) modes, provide widely different predictions for the magnitude of this correction. In this paper we examine the most widely used prescriptions for this reflection coefficient from the point of view of their consistency with the Bohr-van Leeuwen theorem of classical statistical physics, stating that at thermal equilibrium transverse electromagnetic fields decouple from matter in the classical limit. We find that the theorem is satisfied if and only if the TE reflection coefficient vanishes at zero frequency in the classical limit. This criterion appears to rule out some of the models that have been considered recently for describing the thermal correction to the Casimir pressure with non-magnetic metallic plates.

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I. INTRODUCTION

Dispersion forces, frequently called "van der Waals" or "molecular" forces, are long-range electromagnetic forces arising from quantum and thermal charge and current fluctuations existing in microscopic or macroscopic bodies at thermal equilibrium. In view of their pervasive role, from biology to chemistry, from physics to engineering [1], these weak forces have been the subject of intense theoretical and experimental investigations. A distinctive feature of dispersion forces is intimately related to their long-range character. In fact, while short-range forces, like the exchange electromagnetic interaction, are determined by the particular microscopic electronic structure of atoms and molecules, long-range forces display a universal behavior at large distances. This feature is clearly reflected in the famous theory of dispersion forces developed long ago by Lishitz [2], where the particular features of the bodies participating to the interaction, whether atoms or macroscopic bodies, can be fully taken into account by means of their macroscopic permittivities or polarizabilities.

After over fifty years, Lifshitz theory still constitutes the basic theoretical tool universally used by researchers in the field, to interpret the results of modern experiments on dispersion forces. As an example, we mention the numerous recent experiments on the Casimir effect (for a recent review, see [3]), and the beautiful new experiments on the Casimir-Polder interaction of a Bose-Einstein condensate with a dielectric substrate [4]. It is important to note that the precision of the most recent experiments is of a such a level that in order to correctly interpret the data it is now necessary to take into full account a number of small corrections, like temperature effects, the effect of surface corrugations, patch effects etc. (see Ref. [3] for details). In particular, the necessity of careful electrostatic calibrations in precision measurements of the Casimir force has been recently emphasized [5].

In this paper we focus our attention on the influence of material properties of the bodies constituting the system. As remarked above, within Lifshitz theory these properties are fully described by the macroscopic permittivities of the bodies participating in the interaction. As it is well known, the latter quantities are complex functions of the frequency ω of the electromagnetic field, as an effect of dispersive and absorptive properties displayed by all real materials. Now, in Lifshitz theory the free-energy associated with van der Waals forces between condensed bodies is expressed by a sum of terms depending on the reflection coefficients of the plates, evaluated at (imaginary) Matsubara frequencies $\xi_n = 2\pi n k_B T/\hbar$, where k_B is Boltzmann constant, T is the temperature, and the discrete index n in the sum runs form zero to infinity. As a result, evaluation of Lifshitz formula requires knowledge of the reflection coefficients over a wide range of frequencies, extending from zero up to a few times the characteristic frequency $\omega_c = c/(2a)$ of the system, where a is the characteristic separation between the bodies. For typical separations involved in present experiments, ranging from a few tens of nanometers up to a few microns, the characteristic frequency falls somewhere from the IR part of the spectrum, to the near UV. In order to obtain an accurate theoretical prediction for the free-energy, the common practice today is to rely on detailed optical data of the material used in the experiment. The importance of using accurate optical data has been recently emphasized by the authors of Ref. [6], where it is shown that uncertainty in the optical data may easily result in an un-

*Bimonte@na.infn.it%;

certainty of several percent in the evaluation of Lifshitz formula. One point should however be stressed: for the purpose of evaluating the $n = 0$ term of Lifshitz formula, it is always necessary to make an extrapolation of the reflection coefficients to zero frequency. This is a very delicate point, indeed, because the result strongly depends on how the extrapolation is done. Understanding what is the correct extrapolation is crucial in particular for determining the thermal correction to the free-energy in systems involving one or more conductors, because this correction is strongly affected by the magnitude of the $n = 0$ Matsubara term for transverse electric (TE) polarization. This crucial term is determined by the TE reflection coefficients of the plates at zero frequency, and as of now different recipes have been proposed in the literature for this coefficient, resulting in drastically different predictions for the magnitude of the thermal correction. The most popular prescriptions used currently to model real metals can be grouped in two classes: Drude-like models and plasma-like models. The former models [7] are characterized by permittivity functions $\epsilon(\omega)$ displaying at low frequency the same ω^{-1} singularity of the familiar Drude model for the permittivity of a ohmic conductor, resulting in a *vanishing* TE reflection coefficient at zero frequency. On the contrary, plasma-like models are characterized by the ω^{-2} singularity displayed by the plasma model of IR optics, extrapolated to zero frequency. The latter class of models includes in particular the so-called generalized plasma model considered in Ref. [8]. In this model, ohmic dissipation is not accounted for, and only dissipation associated with interband transitions of core electron is considered. The ω^{-2} singularity of plasma-like models entails a *non-vanishing* TE reflection coefficient in the limit of zero frequency, differently from Drude-like models. The different behaviors of the TE reflection coefficient at zero frequency implied by the two classes of models have a dramatic impact on the Casimir force in the limit of large separations between the plates, since Drude-like models lead to a force that is about one-half that implied by plasma-like models. We remark also that in this limit plasma-like models give almost the same force as the simple ideal-metal model of real conductors [3, 9, 10]. At separations between the plates smaller than one micron, where the Casimir force can be measured most accurately, the various prescriptions imply predicted Casimir forces that differ only by a few percents, and it is therefore difficult to distinguish between them by an experiment.

The experimental situation is still unclear, because the present accuracy of Casimir-force measurements is still not sufficient to detect the temperature correction to the Casimir force between two metallic bodies, and indeed there are at present several ongoing and planned experiments to measure it [11]. A recent accurate experiment using a micromechanical oscillator [12] appears to favor the plasma model prescription, but this claim is not yet universally accepted by the community [9]. We remark that as of now there is only one experiment that detected the influence of temperature on the Casimir-Polder interaction of a Bose-Einstein condensate with a dielectric substrate [4]. Clarifying this problem is important also in view of the many experiments on non-newtonian forces at the submicron scale, some of which use metallic surfaces at room temperature [12], making it necessary to estimate accurately the contribution of dispersion forces that must be subtracted from the observed signal. For a lucid description of the problems involved in precision Casimir experiments we address the reader to the recent paper [13].

In this paper we shall examine the low frequency prescriptions for the TE reflection coefficient of insulators and conductors, generally used in investigations of dispersion forces, from the point of view of their compatibility with a very well known theorem of classical solid state physics, namely the Bohr-van Leeuwen theorem [14]. This theorem originated early in the 20th century, in an attempt to explain the absence of strong diamagnetism in normal conductors placed in an external magnetic field. After a simple physical argument due to Bohr, who showed that no net diamagnetic currents can arise in a bounded conductor subjected to a static magnetic field, the theorem was put on a firm theoretical basis by H. J. van Leeuwen. In essence the theorem states that in classical systems at thermal equilibrium matter decouples from the transverse electromagnetic field. It occurred to us that perhaps this theorem could be used to discriminate between existing models used in the current literature to describe dispersion forces. Let us see briefly how the connection arises. One observes that the reflection coefficients of a surface determine the macroscopic response of the surface to an external electromagnetic probe placed outside the surface. It is now the essence of the famous fluctuation dissipation theorem [15] that such response functions are intimately related to equilibrium averages of suitable macroscopic observables of the system. In the case of interest to us, the fluctuation dissipation theorem relates averages of the fluctuating electromagnetic fields outside the surface to its reflection coefficients [18]. This fundamental relation is known to imply a set of general constraints, originating from microscopic reversibility [16], that must be satisfied by the reflection coefficients of any real material, like for example important reciprocity relations [17]. Therefore we were led to wonder if the Bohr-van Leeuwen theorem can be used to put any further constraints on the permitted behavior of the reflection coefficients. We remark that the Casimir and Casimir-Polder interactions are equilibrium phenomena, and therefore they must conform to the principles of equilibrium statistical physics. In order to put this idea to a test we shall evaluate the spectrum of the fluctuating electromagnetic field in the empty space outside one slab, and between two plane-parallel slabs, characterized by a local dielectric response, carefully separating the longitudinal and the transverse components of the electromagnetic field. Having done this, we shall verify whether or not the transverse component of the field decouples

from the slab(s) in the classical limit, as required by the Bohr-van Leeuwen theorem. Interestingly, we shall see that the answer depends exclusively on the behavior of the reflection coefficients of the slab(s), in the limit of zero frequency. In this way we obtain a rather stringent test to decide whether a definite model is admissible or not from the point of view of classical statistical physics. The important result is that the Bohr-van Leeuwen theorem is satisfied if and only if the TE reflection coefficient vanishes at zero frequency, in the classical limit. As a result, we find that Drude-like models are compatible with the Bohr-van Leeuwen theorem, while neither plasma-like models nor the ideal-metal model pass the test. Our conclusions appear to be consistent with the findings of a recent paper [19] which presented a microscopic calculation of the Casimir force between two metallic plates, in the asymptotic limit of large separations between the plates, when the force is dominated by classical thermal fluctuations and the theorem is supposed to apply. It was found there that in this asymptotic regime, the microscopic model assumed in [19] predicts the same Casimir force as the Drude prescription, and therefore it is in disagreement with both plasma-like models and the ideal metal results, which we recall both predict in this limit a force of double magnitude as that implied by Drude-like models.

We point out that in the recent literature on the thermal Casimir effect, another criterion based on statistical physics has been widely considered to discriminate between alternative models for the reflection coefficients of a material slab. This other criterion requires that the Nernst heat theorem be satisfied, in the limit of zero temperature [20]. This alternative criterion leads to conclusions that are not in agreement with what we found on the basis of the Bohr–van Leeuwen criterion, for one finds that the Nernst heat theorem is not satisfied by the Drude prescription in the idealized case of perfect crystals with no defects, but it is satisfied both by the (generalized) plasma models, and by the ideal metal model. In the Drude case, Nernst theorem is however restored if an arbitrarily small amount of impurities are present in the crystal [21]. While we cannot offer a complete resolution of this contradiction, we remark that since the Nernst theorem is intrinsically a quantum result, this criterion is a sense orthogonal to the one proposed in this paper, which is essentially classical. In our judgement, in the absence of a definitive answer, we note that the Bohr–van Leeuwen criterion appears to be more pertinent than the Nernst criterion, for a theoretical assessment of the debated problem of thermal corrections to the Casimir and Casimir-Polder effects at room temperatures, since it is well known that the difficulties posed by this problem are basically of a classical nature (see Sec. V below). In addition, we note that the extrapolation to zero temperature of Lifshitz theory poses very non-trivial problems, associated for example with the possible presence of spatial non-locality (anomalous skin effect [22]). For recent reviews of some theoretical aspects involved by the Nernst heat theorem in the context of Casimir physics, we address the reader to Refs. [9, 23].

The paper is organized as follows. In Section II we review the fluctuation-dissipation theorem, in the context of general linear response theory, while in Section III the theorem is used to derive expressions for the correlators of the electromagnetic fields outside dielectrics and conductors. In Section IV we verify if these correlators satisfy the Bohr- van Leeuwen theorem outside a planar slab, for a number of models of dielectrics and conductors. The case of a plane-parallel cavity, of the type used in Casimir experiments, is considered in Section V. Finally, Section VI contains our conclusions and a discussion of the results.

II. FLUCTUATION-DISSIPATION THEOREM

In this Section we briefly review the principal results of linear response theory, and in particular we present the general fluctuation-dissipation theorem for linear dissipative media. For a review of linear-response theory we address the reader to Refs.[15].

In linear-response theory, one considers a quantum-mechanical system, characterized by a (time-independent) Hamiltonian H_0, in a state of thermal equilibrium described by the density matrix ρ

$$\rho = e^{-\beta H}/\mathrm{tr}(e^{-\beta H}) \;, \tag{1}$$

where $\beta = 1/(k_B T)$. The system is then perturbed by an external perturbation of the form:

$$H_{\rm ext} = -\int d^3\mathbf{r} \sum_j Q_j(\mathbf{r},t)\, f_j(\mathbf{r},t) \tag{2}$$

where $f_j(\mathbf{r},t)$ are external classical forces, and $Q_j(\mathbf{r},t)$ is the dynamical variable of the system conjugate to the force $f_j(\mathbf{r},t)$. One may assume without loss of generality that, in the absence of external forces, the equilibrium values of the quantities $Q_j(\mathbf{r},t)$ all vanish: $\langle Q_j(\mathbf{r},t)\rangle = 0$. The presence of the external forces causes a deviation $\delta\langle Q_i(\mathbf{r},t)\rangle$ of the expectation values of $Q_j(\mathbf{r},t)$ from their equilibrium values. If the forces $f_j(\mathbf{r},t)$ are sufficiently weak, $\delta\langle Q_i(\mathbf{r},t)\rangle$ can be taken to be linear functionals of the applied forces $f_j(\mathbf{r},t)$ according to the formula:

$$\delta\langle Q_i(\mathbf{r},t)\rangle = \sum_j \int d^3\mathbf{r} \int_{-\infty}^{t} dt' \phi_{ij}(\mathbf{r},\mathbf{r}',t-t')\, f_j(\mathbf{r}',t') \;. \tag{3}$$

The above Equation assumes that the system was in equilibrium at $t = -\infty$, and that it reacts to the external force in a causal way. The quantities $\phi_{ij}(\mathbf{r},\mathbf{r}',t-t')$ are called response functions of the system. In principle, they can be measured by applying to the system of interest suitable external classical probes. We now define the admittance $\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',\omega)$ as the (one-sided) Fourier transform

of the response function $\phi_{ij}(\mathbf{r},\mathbf{r}',t)$:

$$\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',\omega)=\int_0^\infty dt\,\phi_{ij}(\mathbf{r},\mathbf{r}',t)\,e^{i\omega t}\;. \tag{4}$$

Being the one-sided transform of a real quantity, the admittance satisfies distinctive analyticity and reality properties. First, it is an analytic function of the complex frequency w in the upper complex plane $\mathcal{C}^+\equiv\{w=\omega+i\delta,\;\delta>0\}$. Second, it satisfies in $\mathcal{C}^+$ the following reality condition

$$\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',-w^*)=\tilde{\phi}^*_{ij}(\mathbf{r},\mathbf{r}',w)\;. \tag{5}$$

The latter property in particular implies that the admittance is real along the imaginary frequency axis $\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',i\xi)=\tilde{\phi}^*_{ij}(\mathbf{r},\mathbf{r}',i\xi)$.

By a straightforward computation in time-dependent perturbation theory one may prove that the response functions $\phi_{ij}(\mathbf{r},\mathbf{r}',t-t')$ are related to the equilibrium expectation values of the commutators of the dynamical variables $Q_i(\mathbf{r},t)$, in the absence of external forces:

$$\phi_{ij}(\mathbf{r},\mathbf{r}',t-t')=\Delta_{ij}(\mathbf{r},\mathbf{r}',t-t')\,\theta(t-t'). \tag{6}$$

Here $\theta(x)$ is Heaviside step function ($\theta(x)=1$ for $x>0$, $\theta(x)=0$ for $x<0$) and

$$\Delta_{ij}(\mathbf{r},\mathbf{r}',t-t')=\frac{i}{\hbar}\,\langle[Q_i(\mathbf{r},t),Q_j(\mathbf{r}',t')]\rangle\;, \tag{7}$$

with $Q_i(\mathbf{r},t)$ the Heisenberg operator:

$$Q_i(\mathbf{r},t)=e^{iH_0t/\hbar}Q_i(\mathbf{r},0)e^{-iH_0t/\hbar}\;. \tag{8}$$

As it is well known, starting from Eq. (6) it is possible to derive several general fluctuation-dissipation theorems, that allow to express the (symmetrized) correlation functions of the quantities $Q_i(\mathbf{r},t)$ in terms of the dissipative component of the response functions ϕ_{ij} [15]. The form of the fluctuation-dissipation theorem of interest to us is expressed by the following relation

$$\int_{-\infty}^\infty dt\,\Delta_{ij}(\mathbf{r},\mathbf{r}',t)\,e^{i\omega t}$$

$$=\frac{i\,\omega}{E_\beta(\omega)}\int_{-\infty}^\infty dt\,\langle\{Q_i(\mathbf{r},t)\,Q_j(\mathbf{r}',0)\}\rangle e^{i\omega t} \tag{9}$$

where

$$E_\beta(\omega)=\frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)$$

is the average free-energy of a quantum oscillator with frequency ω in equilibrium at temperature T, and $\{AB\}=(AB+BA)/2$ denotes the symmetrized product of the operators A and B. If $\Delta_{ij}(\mathbf{r},\mathbf{r}',t)$ has a definite parity under inversion of time, we can easily verify that the l.h.s. of Eq. (9) can be expressed in terms of the admittance. Consider first the case when $\Delta_{ij}(\mathbf{r},\mathbf{r}',t)$ is odd in t. This is the case of interest to us, because the commutators of the electromagnetic fields are indeed odd in time. Then we find:

$$\int_{-\infty}^\infty dt\,\Delta_{ij}(\mathbf{r},\mathbf{r}',t)\,e^{i\omega t}=2\,i\int_0^\infty dt\,\Delta_{ij}(\mathbf{r},\mathbf{r}',t)\,\sin(\omega t)$$

$$=2\,i\,\mathrm{Im}[\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',\omega)]\;. \tag{10}$$

Upon substituting the expression on the second line of Eq. (10) in the l.h.s. of Eq. (9), and after performing the inverse Fourier transform of both sides, we then obtain the following important relation:

$$\langle\{Q_i(\mathbf{r},t)\,Q_j(\mathbf{r}',0)\}\rangle$$

$$=\frac{1}{\pi}\int_{-\infty}^\infty\frac{d\omega}{\omega}\,E_\beta(\omega)\,\mathrm{Im}[\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',\omega)]\,e^{-i\omega t}\;. \tag{11}$$

By further exploiting the fact that, by virtue of Eq. (5), the imaginary part of the admittance is an odd function of ω, we can rewrite the above Equation as:

$$\langle\{Q_i(\mathbf{r},t)\,Q_j(\mathbf{r}',0)\}\rangle$$

$$=\frac{2}{\pi}\int_0^\infty\frac{d\omega}{\omega}\,E_\beta(\omega)\,\mathrm{Im}\,[\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',\omega)]\cos(\omega t)\;. \tag{12}$$

Even if we shall not need it, for completeness we report also the analogous relation that holds if $\Delta_{ij}(\mathbf{r},\mathbf{r}',t)$ is even in t:

$$\langle\{Q_i(\mathbf{r},t)\,Q_j(\mathbf{r}',0)\}\rangle$$

$$=\frac{2}{\pi}\int_0^\infty\frac{d\omega}{\omega}\,E_\beta(\omega)\,\mathrm{Re}\,[\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',\omega)]\sin(\omega t)\;. \tag{13}$$

The above two relations constitute the content of the fluctuation-dissipation theorem. Note in particular that in the odd case the two-times correlation functions of the quantities $Q_i(\mathbf{r},t)$ in Eq. (12) is expressed in terms of the dissipative part of the admittance. It is important to remark that the integrands on the r.h.s. of Eqs. (12) and (13) have no singularity at $\omega=0$, despite the presence of the singular factor ω^{-1}. This so because, from the definition Eq. (4), we see that for vanishing frequency the admittance $\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',0)$ is a finite and real quantity. This ensures that the integrands in Eqs. (12) and (13) both have finite limits as ω approaches zero.

In view of a next use, it is now useful to derive a formula for the equal-time correlators of the quantities $Q_i(\mathbf{r},t)$, in the classical limit. This can be easily done by setting $t=0$ and taking the limit $\hbar\to0$ in the r.h.s. of Eqs. (12) and (13). The only non-trivial case to consider is the odd one, for in the even case we see from Eq. (13)

that the equal-time correlators vanish identically. Then, from Eq. (12) we obtain:

$$\lim_{\hbar\to 0}\langle\{Q_i(\mathbf{r},0)\,Q_j(\mathbf{r}',0)\}\rangle$$

$$=\frac{2k_BT}{\pi}\,\mathrm{Im}\int_0^\infty\frac{d\omega}{\omega}\,\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',\omega)\ . \qquad (14)$$

Assuming that the admittance $\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',w)$ vanishes sufficiently fast at complex infinity in $\mathcal{C}^+$, we can take advantage of analyticity in $\mathcal{C}^+$ of $\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',w)$ to evaluate the integral on the r.h.s., by rotating the integration contour from the real axis to the imaginary one. The rotated contour of integration Γ consists of an infinitesimal arc surrounding the origin in the right sector of $\mathcal{C}^+$, followed by the whole imaginary axis. Since the admittance is real along the imaginary axis, it is clear that the part of the integral over Γ extending along the imaginary axis does not contribute to the r.h.s. of Eq.(14), and therefore we find that the imaginary part of the integral results entirely from the contribution of the infinitesimal arc surrounding the origin. After easy evaluation of the latter contribution, we obtain the simple result:

$$\lim_{\hbar\to 0}\langle\{Q_i(\mathbf{r},0)\,Q_j(\mathbf{r}',0)\}\rangle=k_BT\,\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',0)\ . \qquad (15)$$

We then reach the important conclusion that in the classical limit the equal-time correlators for the quantities $Q_i(\mathbf{r})$ are simply proportional to the zero-frequency limit of the admittance. It is opportune to remark that in the above derivation we have implicitly assumed that the admittance itself is independent of $\hbar$. Obviously when quantum effects contribute to the admittance, the quantity $\tilde{\phi}_{ij}(\mathbf{r},\mathbf{r}',0)$ in the r.h.s. of Eq. (15) must be understood as the classical limit of the admittance at zero-frequency.

III. CORRELATORS OF THE ELECTROMAGNETIC FIELD OUTSIDE DIELECTRICS AND CONDUCTORS

In this Section we shall use the methods described in the previous Section to derive formulae for the correlators of the electromagnetic fields outside dielectrics and conductors. In the spirit of linear response theory, this is done by placing outside the bodies a suitable distribution of classical electric and magnetic dipoles, that work as external probes for the electromagnetic field [18]. In order to separate the longitudinal and the transverse parts of the field, we consider an external Hamiltonian of the following form [28]

$$H_{\rm ext}=\int d^3\mathbf{r}[U(\mathbf{r},t)\rho^{(\rm ext)}(\mathbf{r},t)-\frac{1}{c}\mathbf{A}_\perp(\mathbf{r},t)\cdot\mathbf{j}_\perp^{(\rm ext)}(\mathbf{r},t)]\ , \qquad (16)$$

where $U(\mathbf{r},t)$ and $\mathbf{A}_\perp(\mathbf{r},t)$ are, respectively, the scalar and the transverse vector potentials for the electromagnetic field, while $\rho^{(\rm ext)}(\mathbf{r},t)$ and $\mathbf{j}_\perp^{(\rm ext)}(\mathbf{r},t)$ denote, respectively, external classical distributions of charge and current. Note that $\mathbf{j}_\perp^{(\rm ext)}(\mathbf{r},t)$ is assumed to be transverse:

$$\nabla\cdot\mathbf{j}_\perp^{(\rm ext)}=0\ . \qquad (17)$$

In what follows we shall assume that the bodies constituting the system are non-magnetic ($\mu=1$) dielectrics or conductors with sharp boundaries, characterized each by a frequency dependent electric permittivity $\epsilon(\omega)$. It is moreover assumed that the bodies are homogeneous, in such a way that the permittivities are constant functions of the position within the volume occupied by each body. Under such conditions, it is shown in Ref.[28] that we have two independent sets of response functions for the scalar and the vector potential:

$$U(\mathbf{r},t)=\int_{-\infty}^t dt'\int d^3\mathbf{r}'G(\mathbf{r},\mathbf{r}',t-t')\,\rho^{(\rm ext)}(\mathbf{r}',t')\ , \qquad (18)$$

$$\mathbf{A}_\perp(\mathbf{r},t)=\frac{1}{c}\int_{-\infty}^t dt'\int d^3\mathbf{r}'\mathbf{G}_\perp(\mathbf{r},\mathbf{r}',t-t')\cdot\mathbf{j}_\perp^{(\rm ext)}(\mathbf{r}',t')\ , \qquad (19)$$

where $\mathbf{G}_\perp(\mathbf{r},\mathbf{r}',t-t')$ has to be understood as a dyadic Green function. Recalling that the commutators of the electromagnetic fields are odd functions of time, from the general fluctuation-dissipation theorem Eq. (12) we obtain the following expressions for the correlators of the electromagnetic fields:

$$\langle\{U(\mathbf{r},t)\,U(\mathbf{r}',0)\}\rangle$$

$$=-\frac{2}{\pi}\int_0^\infty\frac{d\omega}{\omega}\,E_\beta(\omega)\,\mathrm{Im}\,[\tilde{G}(\mathbf{r},\mathbf{r}',\omega)]\cos(\omega t)\ , \qquad (20)$$

$$\langle\{U(\mathbf{r},t)\,A_{\perp i}(\mathbf{r}',0)\}\rangle=0\ , \qquad (21)$$

$$\langle\{A_{\perp i}(\mathbf{r},t)\,A_{\perp j}(\mathbf{r}',0)\}\rangle$$

$$=\frac{2}{\pi}\int_0^\infty\frac{d\omega}{\omega}\,E_\beta(\omega)\,\mathrm{Im}\,[\tilde{G}_{\perp ij}(\mathbf{r},\mathbf{r}',\omega)]\cos(\omega t)\ , \qquad (22)$$

where $\tilde{G}(\mathbf{r},\mathbf{r}',\omega)$ and $\tilde{\mathbf{G}}_\perp(\mathbf{r},\mathbf{r}',\omega)$ denote the one-sided Fourier transforms of the Greens functions:

$$\tilde{G}(\mathbf{r},\mathbf{r}',\omega)=\int_0^\infty dt\,G(\mathbf{r},\mathbf{r}',t)\,e^{i\omega t}\ , \qquad (23)$$

$$\tilde{\mathbf{G}}_\perp(\mathbf{r},\mathbf{r}',\omega)=\int_0^\infty dt\,\mathbf{G}_\perp(\mathbf{r},\mathbf{r}',t)\,e^{i\omega t}\ . \qquad (24)$$

The Fourier transforms of the Green's functions can be obtained by solving the following field Equations implied by macroscopic Maxwell Equations:

$$\nabla \cdot [\,\epsilon(\mathbf{r},\omega)\,\nabla \tilde{G}\,] = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')\;, \tag{25}$$

$$(\triangle + \epsilon(\mathbf{r},\omega)\,\omega^2/c^2)\tilde{\mathbf{G}}_{\perp}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\,\delta_{\perp}(\mathbf{r}-\mathbf{r}')\;, \tag{26}$$

where $\delta_{\perp}(\mathbf{r}-\mathbf{r}')$ is the transverse delta-function dyad:

$$\delta^{\perp}_{ij}(\mathbf{x}) = \int d^3\mathbf{k}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{27}$$

with $k = |\mathbf{k}|$. The above field equations must be supplemented by standard boundary conditions at the bodies interfaces, and must be further subjected to the conditions required for retarded Green's functions. The Green's functions $\tilde{G}$ and $\tilde{\mathbf{G}}_{\perp}$ satisfy a number of general properties, that are consequences of microscopic reversibility and of analyticity and reality properties of the permittivity $\epsilon(\omega)$ of any causal material. For a review of these important properties we address the reader to Ref. [28] (and Refs. therein).

For our purposes, it is convenient to split the Green's functions, *outside* the bodies, as sums of an empty-space contribution plus a correction arising from the material bodies:

$$G(\mathbf{r},\mathbf{r}',t-t') = G^{(0)}(\mathbf{r}-\mathbf{r}',t-t') + F^{(\mathrm{mat})}(\mathbf{r},\mathbf{r}',t-t')\;, \tag{28}$$

and

$$\mathbf{G}_{\perp}(\mathbf{r},\mathbf{r}',t-t') = \mathbf{G}^{(0)}_{\perp}(\mathbf{r}-\mathbf{r}',t-t') + \mathbf{F}^{(\mathrm{mat})}_{\perp}(\mathbf{r},\mathbf{r}',t-t')\;. \tag{29}$$

Here, $G^{(0)}$ and $\mathbf{G}^{(0)}_{\perp}$ denote the Green's functions in free space, while $F^{(\mathrm{mat})}$ and $\mathbf{F}^{(\mathrm{mat})}_{\perp}$ describe the effects resulting from the presence of the bodies. Such a splitting presents the advantage that all singularities are included in the free parts $G^{(0)}$ and $\mathbf{G}^{(0)}_{\perp}$, while the quantities $F^{(\mathrm{mat})}$ and $\mathbf{F}^{(\mathrm{mat})}_{\perp}$ are smooth ordinary functions of $\mathbf{r}$ and $\mathbf{r}'$. Upon using Eqs. (28) and (29) into the r.h.s. of Eqs. (20-22) we obtain the following equations for the *changes* of the field correlators outside the material bodies, arising from the presence of the bodies:

$$\delta\langle\{U(\mathbf{r},t)\,U(\mathbf{r}',0)\}\rangle$$

$$= -\frac{2}{\pi}\int_0^{\infty}\frac{d\omega}{\omega}\,E_{\beta}(\omega)\,\mathrm{Im}\,[\tilde{F}^{(\mathrm{mat})}(\mathbf{r},\mathbf{r}',\omega)]\cos(\omega t)\;, \tag{30}$$

$$\delta\langle\{U(\mathbf{r},t)\,A_{\perp i}(\mathbf{r}',0)\}\rangle = 0\;, \tag{31}$$

$$\delta\langle\{A_{\perp i}(\mathbf{r},t)\,A_{\perp j}(\mathbf{r}',0)\}\rangle$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{d\omega}{\omega}\,E_{\beta}(\omega)\,\mathrm{Im}\,[\tilde{F}^{(\mathrm{mat})}_{\perp ij}(\mathbf{r},\mathbf{r}',\omega)]\cos(\omega t)\;, \tag{32}$$

with an obvious meaning for the symbols. It is now easy to derive from the above formulae the expressions for the corresponding changes of the equal-time correlators of the electromagnetic fields that we shall need in the following Sections. For the longitudinal electric field $\mathbf{E}_{\|} = -\nabla U$, from Eq. (30) we obtain

$$\delta\langle\{E_{\| i}(\mathbf{r},0)\,E_{\| j}(\mathbf{r}',0)\}\rangle$$

$$= -\frac{2}{\pi}\int_0^{\infty}\frac{d\omega}{\omega}\,E_{\beta}(\omega)\,\mathrm{Im}\left(\frac{\partial^2\tilde{F}^{(\mathrm{mat})}}{\partial x_i \partial x'_j}\right)\,, \tag{33}$$

For the transverse electric field $\mathbf{E}_{\perp}$ and for the magnetic field $\mathbf{B}$, since $\mathbf{E}_{\perp} = -c^{-1}\,\partial\mathbf{A}_{\perp}/\partial t$, and $\mathbf{B} = \nabla\times\mathbf{A}_{\perp}$, from Eq. (32) we obtain:

$$\delta\langle\{E_{\perp i}(\mathbf{r},0)\,E_{\perp j}(\mathbf{r}',0)\}\rangle$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{d\omega}{\omega}\,E_{\beta}(\omega)\,k_0^2\,\mathrm{Im}\,[\,\tilde{F}^{(\mathrm{mat})}_{\perp ij}(\mathbf{r},\mathbf{r}',\omega)]\;, \tag{34}$$

where $k_0 = \omega/c$ and

$$\delta\langle\{B_i(\mathbf{r},0)\,B_j(\mathbf{r}',0)\}\rangle$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{d\omega}{\omega}\,E_{\beta}(\omega)\,\mathrm{Im}\,[\,(\overrightarrow{\nabla}_{\mathbf{r}}\times\tilde{\mathbf{F}}^{(\mathrm{mat})}_{\perp}\times\overleftarrow{\nabla}_{\mathbf{r}'})_{ij}]\;. \tag{35}$$

Finally, from Eq. (15) we obtain the following equations for the matter contributions to the equal-time electric and magnetic correlators in the classical limit:

$$\lim_{\hbar\to 0}\,\delta\langle\{E_{\perp i}(\mathbf{r},0)\,E_{\perp j}(\mathbf{r}',0)\}\rangle = k_B\,T\,\mathcal{E}^{(\mathrm{cl})}_{\perp ij}(\mathbf{r},\mathbf{r}')\;, \tag{36}$$

and

$$\lim_{\hbar\to 0}\,\delta\langle\{B_{\perp i}(\mathbf{r},0)\,B_{\perp j}(\mathbf{r}',0)\}\rangle = k_B\,T\,\mathcal{B}^{(\mathrm{cl})}_{\perp ij}(\mathbf{r},\mathbf{r}')\;, \tag{37}$$

where we defined

$$\mathcal{E}^{(\mathrm{cl})}_{\perp ij}(\mathbf{r},\mathbf{r}') = \lim_{\omega\to 0}(k_0^2\;\tilde{F}^{(\mathrm{mat})}_{\perp ij}(\mathbf{r},\mathbf{r}',\omega))\,, \tag{38}$$

and

$$\mathcal{B}^{(\mathrm{cl})}_{\perp ij}(\mathbf{r},\mathbf{r}') = \lim_{\omega\to 0}\,(\overrightarrow{\nabla}_{\mathbf{r}}\times\tilde{\mathbf{F}}^{(\mathrm{mat})}_{\perp}\times\overleftarrow{\nabla}_{\mathbf{r}'})_{ij}\;. \tag{39}$$

According to the Bohr-van Leeuwen theorem [14] the transverse electromagnetic fields decouples from matter in the classical limit. From Eqs. (36) and (37) we see that the theorem is fulfilled if and only if the quantities $\mathcal{E}^{(\mathrm{cl})}_{\perp ij}$ and $\mathcal{B}^{(\mathrm{cl})}_{\perp ij}$ all vanish identically outside the bodies. Equipped with these formulae, we are ready now to examine whether the theorem is satisfied in the simple case of a dielectric slab.

IV. THE BOHR-VAN LEEUWEN THEOREM FOR ONE DIELECTRIC OR CONDUCTING SLAB

In this Section we use the results of the previous Sections to verify whether the Bohr-van Leeuwen theorem is satisfied outside a plane-parallel dielectric or conducting slab characterized by a spatially local permittivity $\epsilon(\omega)$. We choose our cartesian coordinate system in such a way that the z axis is perpendicular to the slab surface, with the slab occupying the $z<0$ half-space.

The relevant Green's functions for this problem have already been worked out in Ref. [28]. The matter contribution $\tilde{\mathbf{F}}_{\perp}^{(\rm wall)}$ to the tensor Green's functions was found to have the following form:

$$\tilde{F}_{\perp ij}^{(\rm wall)}=\tilde{U}_{ij}^{(\rm wall)}+\tilde{V}_{ij}^{(\rm wall)}\;. \tag{40}$$

In this equation, $\tilde{U}_{ij}^{(\rm wall)}$ is the quantity

$$\tilde{U}_{ij}^{(\rm wall)}=i\int\frac{d^2\mathbf{k}_{\perp}}{2\pi\,k_z}\left(e_{\perp i}e_{\perp j}\,r^{(s)}(\omega,k_{\perp})\right.$$

$$\left.+\frac{\xi_i^{(+)}\xi_j^{(-)}}{k_0^2}\,r^{(p)}(\omega,k_{\perp})\right)\,e^{i\mathbf{k}^{(+)}\cdot\mathbf{r}-i\mathbf{k}^{(-)}\cdot\mathbf{r}'}\;, \tag{41}$$

where $\mathbf{k}_{\perp}$ denotes the projection of the wave-vector onto the (x,y) plane, and we have defined $k_z=\sqrt{k_0^2-k_{\perp}^2}$ (the square root is defined such that $\mathrm{Im}(k_z)>0$), $\mathbf{e}_{\perp}=\hat{\mathbf{z}}\times\hat{\mathbf{k}}_{\perp}$, $\mathbf{k}^{(\pm)}=\mathbf{k}_{\perp}\pm k_z\hat{\mathbf{z}}$ and $\xi^{\pm}=k_{\perp}\hat{\mathbf{z}}\mp k_z\hat{\mathbf{k}}_{\perp}$, while $r^{(s)}(\omega,\mathbf{k}_{\perp})$ and $r^{(p)}(\omega,\mathbf{k}_{\perp})$ are the familiar Fresnel reflections coefficients for TE and TM waves, respectively:

$$r^{(s)}(\omega,k_{\perp})=\frac{k_z-s}{k_z+s}\;, \tag{42}$$

$$r^{(p)}(\omega,k_{\perp})=\frac{\epsilon(\omega)\,k_z-s}{\epsilon(\omega)\,k_z+s}\;, \tag{43}$$

where $s=\sqrt{\epsilon(\omega)k_0^2-k_{\perp}^2}$, and again the square root is defined such that $\mathrm{Im}(s)>0$. For the quantity $\tilde{V}_{ij}^{(\rm wall)}$ we have

$$\tilde{V}_{ij}^{(\rm wall)}=\frac{1}{k_0^2}\frac{\partial^2\tilde{\Psi}^{(\rm wall)}}{\partial x_i\partial x'_j} \tag{44}$$

where

$$\tilde{\Psi}^{(\rm wall)}=-\bar{r}(\omega)\int\frac{d^2\mathbf{k}_{\perp}}{2\pi\,k_{\perp}}\,e^{i\bar{\mathbf{k}}^{(+)}\cdot\mathbf{r}-i\bar{\mathbf{k}}^{(-)}\mathbf{r}'}\;,\quad z\geq 0\,, \tag{45}$$

where $\bar{\mathbf{k}}^{(\pm)}=\mathbf{k}_{\perp}\pm ik_{\perp}\,\hat{\mathbf{z}}$ and the reflection coefficient $\bar{r}(\omega)$ is

$$\bar{r}(\omega)=\frac{\epsilon(\omega)-1}{\epsilon(\omega)+1}\,. \tag{46}$$

We can now evaluate the quantities $\mathcal{E}_{\perp ij}^{(\rm cl)}$ and $\mathcal{B}_{\perp ij}^{(\rm cl)}$ defined in Eqs. (38) and (39). Using the following relations

$$\xi^{(\pm)}=\mp i\,\bar{\mathbf{k}}^{(\pm)}+O(\omega^2)\;, \tag{47}$$

$$\mathbf{k}^{(\pm)}=\bar{\mathbf{k}}^{(\pm)}+O(\omega^2)\;, \tag{48}$$

$$k_z=i\,k_{\perp}+O(\omega^2)\;, \tag{49}$$

and observing that $\mathbf{k}^{(\pm)}\times\mathbf{e}_{\perp}=\xi^{(\pm)}$ and $\xi^{(\pm)}\times\mathbf{k}^{(\pm)}=k_0^2\,\mathbf{e}_{\perp}$ it is easy to verify that

$$\mathcal{E}_{\perp ij}^{(\rm cl)}=\lim_{\omega\to 0}\int\frac{d^2\mathbf{k}_{\perp}}{2\pi\,k_{\perp}}\left\{k_0^2\,r^{(s)}(\omega,k_{\perp})\,e_{\perp i}e_{\perp j}\right.$$

$$\left.+\left[r^{(p)}(\omega,k_{\perp})-\bar{r}(\omega)\right]\bar{k}_i^{(+)}\bar{k}_j^{(-)}\right\}e^{i\bar{\mathbf{k}}^{(+)}\cdot\mathbf{r}-i\bar{\mathbf{k}}^{(-)}\mathbf{r}'}\;, \tag{50}$$

$$\mathcal{B}_{\perp ij}^{(\rm cl)}=\lim_{\omega\to 0}\int\frac{d^2\mathbf{k}_{\perp}}{2\pi\,k_{\perp}}\left\{r^{(s)}(\omega,k_{\perp})\,\bar{k}_i^{(+)}\bar{k}_j^{(-)}\right.$$

$$\left.+k_0^2\,r^{(p)}(\omega,k_{\perp})\,e_{\perp i}e_{\perp j}\right\}e^{i\bar{\mathbf{k}}^{(+)}\cdot\mathbf{r}-i\bar{\mathbf{k}}^{(-)}\mathbf{r}'}\,. \tag{51}$$

As we see from the above two Equations, whether the Bohr-van Leeuwen theorem is satisfied or not depends entirely on the behavior of the reflection coefficients, or what is the same of the electric permittivity $\epsilon(\omega)$ of the slab, in the limit of zero frequency. We consider now several classes of models for the low-frequency behavior of the permittivity of the materials that are usually used in present experiments on dispersion forces [3]. First we have insulators, whose permittivities approach a finite limit at zero frequency:

$$\epsilon(\omega)=\epsilon_0+O(\omega)\qquad(\text{insulator})\,. \tag{52}$$

Then we have normal (i.e. non superconducting) non-magnetic ohmic conductors. For these materials several distinct models have been considered in the current literature on dispersion forces, and we classify them generally as Drude-like models, plasma-like models and the ideal metal model. By Drude-like models we mean any models displaying at low frequency the same singular behavior of the familiar Drude-model, characterized by an ω^{-1} singularity:

$$\epsilon(\omega)=\frac{4\pi i\sigma_0}{\omega}+O(1);\quad(\text{Drude}-\text{like models})\;, \tag{53}$$

where σ_0 is the dc conductivity. By plasma-like models we mean instead any models characterized by the same ω^{-2} singularity displayed by the familiar plasma-model of IR optics

$$\epsilon(\omega)=-\frac{\Omega_P^2}{\omega^2}+O(\omega^{-1})\qquad(\text{plasma}-\text{like models})\;, \tag{54}$$

where Ω_P is the plasma frequency. These models include in particular the so-called generalized plasma model considered recently in connection with the Casimir effect [8]. Finally we have the ideal-metal model which is better formulated directly in terms of the reflection coefficients:

$$r^{(p)} = \bar{r} = -r^{(s)} = 1 \quad \text{(ideal metal)} . \tag{55}$$

We now estimate the quantities $\mathcal{E}^{(\mathrm{cl})}_{\perp ij}$ and $\mathcal{B}^{(\mathrm{cl})}_{\perp ij}$ using the above models. It is a simple matter to verify that for all models, the reflection coefficients $r^{(p)}$, $r^{(s)}$ and $\bar{r}$ are finite in the limit $\omega \to 0$. It is also possible to check (see also Appendix B of Ref. [28]) that the difference $r^{(p)}(\omega, k_\perp) - \bar{r}(\omega)$ occurring in the r.h.s. of Eq. (50), vanishes always at zero frequency (in the ideal case it is identically zero). In view of this, we see from Eq. (50) that the quantities $\mathcal{E}^{(\mathrm{cl})}_{\perp ij}$ vanish for all models:

$$\mathcal{E}^{(\mathrm{cl})}_{\perp ij} = 0 . \tag{56}$$

However, in the case of $\mathcal{B}^{(\mathrm{cl})}_{\perp ij}$ we see from Eq. (51) that only the second term between the curly brackets vanishes always in the limit of zero frequency, leaving us with:

$$\mathcal{B}^{(\mathrm{cl})}_{\perp ij} = \int \frac{d^2\mathbf{k}_\perp}{2\pi\, k_\perp}\, r^{(s)}(0, k_\perp)\, \bar{k}^{(+)}_i \bar{k}^{(-)}_j e^{i\bar{\mathbf{k}}^{(+)}\cdot\mathbf{r} - i\bar{\mathbf{k}}^{(-)}\mathbf{r}'} , \tag{57}$$

where we set

$$r^{(s)}(0, k_\perp) \equiv \lim_{\omega\to 0} r^{(s)}(\omega, k_\perp) . \tag{58}$$

This equation shows that the key quantity to consider is the zero-frequency limit of the TE reflection coefficient, for we see that the Bohr-van Leeuwen is satisfied if and only if $r^{(s)}(0, k_\perp)$ vanishes. Now, it is easily seen that the insulator model and Drude-like models both imply a vanishing value for $r^{(s)}(0, k_\perp)$:

$$r^{(s)}(0, k_\perp) = 0 \quad \text{(insul. and Drude} - \text{like models)}, \tag{59}$$

On the contrary, with plasma-like models we find

$$r^{(s)}(0, k_\perp) = \frac{k_\perp - \sqrt{k_\perp^2 + \Omega_P^2/c^2}}{k_\perp + \sqrt{k_\perp^2 + \Omega_P^2/c^2}} \quad \text{(plasma} - \text{like models)} . \tag{60}$$

For the ideal-metal model we obviously have

$$r^{(s)}(0, k_\perp) = -1 \quad \text{(ideal metal)} . \tag{61}$$

In view of these formulae, we reach the important conclusion that both the dielectric and the ohmic (or Drude) models are consistent with the Bohr-van Leeuwen theorem, while both plasma-like models and the ideal model are not. It is interesting to remark that even when the latter two models are considered, the quantity $\mathcal{B}^{(\mathrm{cl})}_{\perp ij}$ becomes negligible at large distances from the slab, and therefore the inconsistency revealed here is expected to be important only in the study of proximity effects like the Casimir effect to be considered in the next Section.

To avoid misunderstandings, we should repeat the warning made at the end Section II. We are tacitly admitting here that the reflection coefficient $r^{(s)}(0, k_\perp)$ is independent of $\hbar$. Obviously, when quantum effects are important, it is understood that in Eq. (57) one should take the classical limit of $r^{(s)}(0, k_\perp)$. This consideration applies for example both to magnetic materials, and especially so to superconductors [24].

V. THE BOHR-VAN LEEUWEN THEOREM AND THE THERMAL CASIMIR EFFECT

In this Section, we shall discuss the consequences of the Bohr-van Leeuwen theorem for the much debated problem of the thermal Casimir effect in metallic systems. Therefore, we consider the Casimir apparatus consisting of two non-magnetic plane-parallel slabs, separated by a vacuum gap of width d. We let $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ the permittivities of the two slabs, which are assumed to occupy, respectively, the regions $z \leq 0$ and $z \geq d$.

The Green's functions for this problem were worked out in Ref. [28], and we can use them here. The material contribution $\tilde{\mathbf{F}}^{(\mathrm{cav})}_\perp$ to the transverse Green's function was found to be of the form:

$$\tilde{\mathbf{F}}^{(\mathrm{cav})}_\perp = \tilde{\mathbf{U}}^{(\mathrm{cav})} + \tilde{\mathbf{V}}^{(\mathrm{cav})} . \tag{62}$$

Here

$$\tilde{U}^{(\mathrm{cav})}_{ij} = i \int \frac{d^2\mathbf{k}_\perp}{2\pi k_z} \left\{ \left[\left(\frac{1}{\mathcal{A}_s} - 1 \right) \left(e^{i\mathbf{k}^{(+)}\cdot(\mathbf{r}-\mathbf{r}')} + e^{i\mathbf{k}^{(-)}\cdot(\mathbf{r}-\mathbf{r}')} \right) + \frac{r_1^{(s)}}{\mathcal{A}_s} e^{i\mathbf{k}^{(+)}\cdot\mathbf{r} - i\mathbf{k}^{(-)}\cdot\mathbf{r}'} + \frac{r_2^{(s)}}{\mathcal{A}_s} e^{i\mathbf{k}^{(-)}\cdot\mathbf{r} - i\mathbf{k}^{(+)}\cdot\mathbf{r}' + 2ik_z d} \right] e_{\perp i} e_{\perp j} \right.$$

$$\left. + \frac{1}{k_0^2} \left[\left(\frac{1}{\mathcal{A}_p} - 1 \right) \left(\xi^{(+)}_i \xi^{(+)}_j e^{i\mathbf{k}^{(+)}\cdot(\mathbf{r}-\mathbf{r}')} + \xi^{(-)}_i \xi^{(-)}_j e^{i\mathbf{k}^{(-)}\cdot(\mathbf{r}-\mathbf{r}')} \right) + \xi^{(+)}_i \xi^{(-)}_j \frac{r_1^{(p)}}{\mathcal{A}_p} e^{i\mathbf{k}^{(+)}\cdot\mathbf{r} - i\mathbf{k}^{(-)}\mathbf{r}'} \right. \right.$$

$$+\xi_i^{(-)}\xi_j^{(+)}\,\frac{r_2^{(p)}}{\mathcal{A}_p}\,e^{i\mathbf{k}^{(-)}\cdot\mathbf{r}-i\mathbf{k}^{(+)}\mathbf{r}'+2ik_zd}\Bigg]\Bigg\}\ , \tag{63}$$

where $r_i^{(\alpha)}$, $\alpha = s,p$ are the Fresnel reflection coefficients of slab i for polarization α, and $\mathcal{A}_\alpha = 1 - r_1^{(\alpha)} r_2^{(\alpha)}\ \exp(2ik_zd)$. As to $\tilde{\mathbf{V}}^{(\mathrm{cav})}$ it has the expression:

$$\tilde{V}_{ij}^{(\mathrm{cav})} = \frac{1}{k_0^2}\frac{\partial^2 \Psi^{(\mathrm{cav})}}{\partial x_i \partial x'_j}\ , \tag{64}$$

with

$$\tilde{\Psi}^{(\mathrm{cav})} = \int \frac{d^2\mathbf{k}_\perp}{2\pi\, k_\perp}\left[\left(\frac{1}{\mathcal{A}}-1\right)\left(e^{i\bar{\mathbf{k}}^{(+)}\cdot(\mathbf{r}-\mathbf{r}')}+e^{i\bar{\mathbf{k}}^{(-)}\cdot(\mathbf{r}-\mathbf{r}')}\right)\right.$$

$$\left.-\frac{1}{\mathcal{A}}\left(\bar{r}_1\,e^{i\bar{\mathbf{k}}^{(+)}\cdot\mathbf{r}-i\bar{\mathbf{k}}^{(-)}\cdot\mathbf{r}'}+\bar{r}_2\,e^{i\bar{\mathbf{k}}^{(-)}\cdot\mathbf{r}-i\bar{\mathbf{k}}^{(+)}\cdot\mathbf{r}'-2k_\perp d}\right)\right]\ , \tag{65}$$

where $\mathcal{A} = 1 - \bar{r}_1(\omega)\,\bar{r}_2(\omega)\ \exp(-2k_\perp\, d)$. Concerning the matter contribution to the scalar Green's function $\tilde{F}^{(\mathrm{cav})}$, in Ref. [28] it was shown that

$$\tilde{F}^{(\mathrm{cav})} = \tilde{\Psi}^{(\mathrm{cav})}\ . \tag{66}$$

As it is well known [2] the Casimir pressure acting on the slabs is given by the thermal average of the zz component $\langle T_{zz}^{(\mathrm{mat})}\rangle$ of the matter contribution to the Maxwell stress-tensor. Since, according to Eq. (31) the scalar potential and the vector potential are uncorrelated, $\langle T_{zz}^{(\mathrm{mat})}\rangle$ decomposes into the sum of a longitudinal and a transverse contributions:

$$\langle T_{zz}^{(\mathrm{mat})}\rangle = \langle T_{\parallel zz}^{(\mathrm{mat})}\rangle + \langle T_{\perp zz}^{(\mathrm{mat})}\rangle\ , \tag{67}$$

and it is interesting to evaluate the two contributions separately. Upon recalling the classical expression of the Maxwell stress tensor in vacuum:

$$T_{ij} = \frac{1}{4\pi}\left(\frac{1}{2}\delta_{ij}E_kE_k - E_i\,E_j + \frac{1}{2}\delta_{ij}B_kB_k - B_i\,B_j\right)\ , \tag{68}$$

we find

$$\langle T_{\parallel zz}^{(\mathrm{mat})}\rangle = \frac{1}{8\pi}\sum_k \lambda_k\,\delta\langle\{E_{\parallel k}(\mathbf{r},0)\,E_{\parallel k}(\mathbf{r},0)\}\rangle\ , \tag{69}$$

and

$$\langle T_{\perp zz}^{(\mathrm{mat})}\rangle = \frac{1}{8\pi}\sum_k \lambda_k\ [\delta\langle\{E_{\perp k}(\mathbf{r},0)\,E_{\perp k}(\mathbf{r},0)\}\rangle$$

$$+\,\delta\langle\{B_k(\mathbf{r},0)\,B_k(\mathbf{r},0)\}\rangle]\ , \tag{70}$$

where $\lambda_1 = \lambda_2 = -\lambda_3 = 1$. By recalling Eqs. (33), (34) and (35) and using the explicit expression of $\tilde{F}^{(\mathrm{cav})}$ and $\tilde{F}_{\perp ij}^{(\mathrm{cav})}$ given above, after some elementary algebra one obtains the following expressions for $\langle T_{\parallel zz}^{(\mathrm{mat})}\rangle$ and $\langle T_{\perp zz}^{(\mathrm{mat})}\rangle$:

$$\langle T_{\parallel zz}^{(\mathrm{mat})}\rangle = \frac{1}{\pi^2}\int_0^\infty \frac{d\omega}{\omega}E_\beta(\omega)\int dk_\perp\, k_\perp^2$$

$$\times\ \mathrm{Im}\left[\left(1-\frac{e^{2k_\perp d}}{\bar{r}_1\bar{r}_2}\right)^{-1}\right]\ , \tag{71}$$

$$\langle T_{\perp zz}^{(\mathrm{mat})}\rangle = -\frac{2}{\pi}\int_0^\infty \frac{d\omega}{\omega}E_\beta(\omega)\,\mathrm{Im}\left[\mathcal{T}_\perp(\omega)\right] \tag{72}$$

where

$$\mathcal{T}_\perp(\omega) = \frac{1}{2\pi}\int_0^\infty dk_\perp\, k_\perp\left\{q\sum_{\alpha=s,p}\left(\frac{e^{-2ik_zd}}{r_1^{(\alpha)}r_2^{(\alpha)}}-1\right)^{-1}\right.$$

$$\left.-\,k_\perp\left(\frac{e^{2k_\perp d}}{\bar{r}_1\bar{r}_2}-1\right)^{-1}\right\}\ , \tag{73}$$

with $q = -ik_z$. We note that when we take the sum of $\langle T_{\parallel zz}^{(\mathrm{mat})}\rangle$ and $\langle T_{\perp zz}^{(\mathrm{mat})}\rangle$, the longitudinal contribution $\langle T_{\parallel zz}^{(\mathrm{mat})}\rangle$ cancels against the transverse contribution resulting from the second term between the curly brackets on the r.h.s. of Eq. (73). The resulting expression for $\langle T_{zz}^{(\mathrm{mat})}\rangle$ then reproduces the following well known Lifshitz formula for the Casimir pressure $P(d,T)$, expressed as an integral over real frequencies [2]:

$$P(d,T) = -\frac{1}{\pi^2}\int_0^\infty \frac{d\omega}{\omega}E_\beta(\omega)\ \int_0^\infty dk_\perp\, k_\perp$$

$$\times\ \mathrm{Im}\left\{q\sum_{\alpha=s,p}\left(\frac{e^{-2ik_zd}}{r_1^{(\alpha)}r_2^{(\alpha)}}-1\right)^{-1}\right\}\ . \tag{74}$$

We can now verify whether the transverse contribution $\langle T_{\perp zz}^{(\mathrm{mat})}\rangle$ vanishes in the classical limit, as required by the Bohr-van Leeuwen theorem. By taking the limit of the r.h.s. of Eq. (72) for $\hbar \to 0$ we obtain:

$$\lim_{\hbar\to 0}\langle T_{\perp zz}^{(\mathrm{mat})}\rangle = -\frac{2k_BT}{\pi}\,\mathrm{Im}\int_0^\infty \frac{d\omega}{\omega}\,\mathcal{T}_\perp(\omega)\ . \tag{75}$$

Evaluation of the integral on the r.h.s. is made easy after we note that the quantity $\mathcal{T}_\perp(\omega)$ vanishes for large

frequencies, and is an analytic function of the frequency in the upper complex half-plane $\mathcal{C}^+$, as a result of analyticity properties of the reflection coefficients. These properties permit us to rotate the contour of integration from the real axis towards the imaginary axis, as we did already in Section II. Since along the imaginary axis $\mathcal{T}_\perp(w)$ is real (because the reflection coefficients are real for imaginary frequencies), one finds that the integral on the r.h.s. of Eq. (75) receives its only contribution from the pole in the origin. The latter is easily evaluated, giving us the result:

$$\lim_{\hbar\to 0}\langle T^{(\mathrm{mat})}_{\perp zz}\rangle = -\,k_B T \lim_{\omega\to 0} \mathcal{T}_\perp(\omega)\,. \qquad (76)$$

We then reach again the conclusion that validity of the Bohr- van Leeuwen theorem depends on the behavior of the reflection coefficients, or what is the same of the permittivities, in the limit of zero frequency. At this point we assume for simplicity that the slabs are made of the same material, in such a way that $\epsilon_1(\omega)=\epsilon_2(\omega)=\epsilon(\omega)$, and we consider again the four models Eqs. (52-55) for the permittivity in the low-frequency limit. It is easy to check that for all these models the quantity:

$$q\left(\frac{e^{-2ik_z d}}{(r^{(p)})^2}-1\right)^{-1} -\, k_\perp\left(\frac{e^{2k_\perp d}}{\bar{r}^2}-1\right)^{-1} \qquad (77)$$

occurring on the r.h.s. of Eq. (73), vanishes like ω^2, and therefore it does not contribute to $\mathcal{T}_\perp$ in the limit of vanishing frequencies. In view of this, only TE-modes may give a contribution to $\mathcal{T}_\perp$ in the limit of vanishing frequencies and from Eq. (73) we obtain:

$$\lim_{\hbar\to 0}\langle T^{(\mathrm{mat})}_{\perp zz}\rangle = -\frac{k_B T}{2\pi}\int_0^\infty dk_\perp\, k_\perp^2$$

$$\times\left(\frac{e^{2k_\perp d}}{(r^{(s)}(0,k_\perp))^2}-1\right)^{-1}. \qquad (78)$$

The Bohr-van Leeuwen theorem requires that the quantity on the r.h.s. vanishes for all separations, and this is only possible if $r^{(s)}(0,k_\perp)$ vanishes. Recalling the values for TE reflection coefficient $r^{(s)}(0,k_\perp)$ listed in Eqs. (59-61), we see again that only the insulator and Drude-like models are consistent with the Bohr- van Leeuwen theorem, while plasma-like models and the ideal model are not.

As it is well known after rotation of the frequency domain of integration from the real axis to the imaginary axis, Lifshitz formula Eq. (74) takes the form of a sum over so-called imaginary Matsubara frequencies:

$$P(d,T) = -\frac{k_B T}{\pi}\sum_{n=0}^\infty\left(1-\frac{1}{2}\delta_{n0}\right)\int_0^\infty dk_\perp\, k_\perp\, q_n$$

$$\times\sum_{\alpha=s,p}\left(\frac{e^{2q_n d}}{r_1^{(\alpha)}(i\xi_n,k_\perp)\,r_2^{(\alpha)}(i\xi_n,k_\perp)}-1\right)^{-1}, \qquad (79)$$

where $\xi_n = 2\pi n k_B T/\hbar$ are the Matsubara frequencies and $q_n=\sqrt{k_\perp^2+\xi_n^2/c^2}$. By comparing the r.h.s. of Eq. (78) with the r.h.s. of Lifshitz formula Eq. (79) we see that the former quantity coincides with the $n=0$ contribution to the Casimir pressure for TE polarization. As it is well known this very term has been the object of a long debate in the recent literature on the thermal Casimir effect, and as of now there is no consensus among experts on its actual magnitude in the case of normal metallic plates [3, 9, 10]. The analysis carried here shows that the Bohr-van Leeuwen theorem requires that this term be zero for normal non magnetic conductors. Quantum effects permit of course non vanishing values for the $n=0$ TE contribution, without violating the theorem, for example in magnetic materials and especially in superconductors [24] (see remarks at the end of the previous Section).

VI. CONCLUSIONS AND DISCUSSION

The problem of the thermal correction to the Casimir and Casimir-Polder interactions in systems involving normal ohmic conductors has attracted considerable attention in the recent literature on dispersion forces. Despite numerous theoretical and experimental investigations, the resolution of this problem is not clear yet.

On the theoretical side, several distinct models have been proposed, that give widely different predictions for the magnitude of this correction. The difficulty is that in order to get a definite value for the thermal correction, it is necessary to make a definite extrapolation to zero frequency of the optical data of the plates, because the theoretical prediction is very sensitive to the limiting behavior of the reflection coefficients for vanishing frequency. Unfortunately, this extrapolation cannot be done solely on the basis of direct measurements, and it unavoidably requires making some theoretical assumptions. Different ansatz have been proposed in the literature, each supported by some arguments, that give widely different predictions for the magnitude of the thermal correction.

The experimental situation is not definite either. In principle, the best way to clarify the issue would be to measure the Casimir force between two metallic plates at separations of a few microns, because for such large separations different theoretical models predict forces that differ by as much as about fifty percent. Unfortunately, at these distances the Casimir force is very small, and at the moment no one has been able to measure it with sufficient precision [11]. Below one micron, where the Casimir force can be measured most accurately, the thermal correction gets very small and therefore it is difficult to observe it. At the moment of this writing, the most relevant experiments in this regard are those at Purdue University [12] (see also Ref.[3]). While the achieved precision is not sufficient to detect the thermal correction predicted by certain theoretical models (of the plasma

type), the authors claimed that their measurements rule out at high confidence level the much larger thermal correction predicted by alternative theoretical models (of ohmic conductor type). As of now, no other experimental groups have carried out measurements of the thermal correction to the Casimir force. For a definitive assessment, it would be desirable to have more experiments, possibly of different types. Perhaps, further insights may come from superconductors [24].

In this paper we have examined the most widely used models for the low-frequency behavior of the reflection coefficients of dielectrics and conductors, from the point of view of their consistency with the Bohr- van Leeuwen theorem [14]. This is a well known theorem in classical statistical physics, stating that at thermal equilibrium the transverse electromagnetic field decouples from matter. As it is well known, this theorem provides the basic explanation for the absence of strong diamagnetism in normal conductors. The theorem has a very general character, and it fact it also holds under the less restrictive assumption of local kinetic equilibrium [25]. In this paper, we evaluated the correlation functions for the transverse electromagnetic field outside a dielectric or conducting slab, and inside a planar cavity of the type used in Casimir experiments. Upon taking the classical limit of these correlators, we found that decoupling of the transverse electromagnetic field occurs if and only if the reflection coefficient for transverse electric (TE) modes vanishes at zero frequency, in the classical limit. According to this result, we conclude that the dielectric and the ohmic conductor models are completely consistent with the Bohr- van Leeuwen theorem, while neither plasma-like models nor the ideal metal model are. Interestingly, in the case of a cavity, the average value of the Maxwell stress tensor, providing the Casimir pressure, is consistent with the theorem if and only if the $n = 0$ Matsubara mode for TE polarization gives no contribution to the Casimir pressure. This term is precisely the source of the present controversies on the thermal Casimir effect for metallic plates. From the point of view of the Bohr- van Leeuwen theorem the answer is clear: this term must be zero, up to quantum effects. We remark that non-vanishing values for the TE reflection coefficient are of course possible in materials displaying a magnetic response, and in superconductors, since both phenomena arise from quantum effects that disappear in the classical limit. When such materials are considered a on-vanishing contribution to the Casimir pressure from the $n = 0$ Matsubara mode is clearly possible, without violating the Bohr- van Leeuwen theorem.

The reader may be disturbed by hearing that the familiar ideal-metal model is inconsistent with the Bohr- van Leeuwen theorem. After all, this is the model that Casimir himself used to investigate the effect that goes under his name, sixty years ago [26]. We remark that this model constitutes an extreme idealization of real metals, and therefore one should not expect it to be universally valid. A hint that this model may provide incorrect answers in problems involving proximity effects of the quantized electromagnetic field near material surfaces was pointed out long ago by Milonni [27] who found that ideal-metal boundary conditions imply a violation of equal-time canonical commutation relations for the transverse electromagnetic field in the vicinity of a metallic surface. In a recent work [28], the author of the present paper showed that such a violation is an artifact of the idealized boundary conditions for a perfect metal. Indeed, if account is taken of dispersive and dissipative features of real materials, no such violation is found, and the canonical commutation relations are restored at all point outside the conductor. As to the range of validity of the ideal-metal model, we have seen above that the field correlators derived from this model violate the Bohr- van Leeuwen theorem only at finite temperature and only in the vicinity of the metallic surface. We infer from this that the model has to be considered unreliable in the study of the proximity effects like the Casimir and the Casimir-Polder interactions, in situations where temperature effects become important.

Finally, we note that in the recent literature on the Casimir effect there has been a lively debate on the influence of free charge carriers that always exist in poor conductors and in semiconductors [3, 29]. The Bohr- van Leeuwen theorem cannot give any help to discriminate between the alternative models that have been proposed for these materials, because in this case differences in the predicted magnitudes of the Casimir and Casimir-Polder interactions arise either from longitudinal fields or from TM fields. As far as longitudinal fields are concerned, the Bohr- van Leeuwen theorem obviously does not apply. In the case of TM electromagnetic fields, we have seen in Sections IV and V that they always decouple in the classical limit, whatever model is used for the dielectric function.

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[1] V. A. Parsegian, *Van der Waals Forces* (Cambridge University Press, Cambridge, England, 2005).

[2] E.M. Lifshitz, Sov. Phys. JETP **2**, 73 (1956); E.M. Lifshitz and L.P. Pitaevskii, *Landau and Lifshitz Course of Theoretical Physics: Statistical Physics Part II* (Butterworth-Heinemann, 1980).

[3] G.L. Klimchitskaya, U. Mohideen and V.M. Mostepanenko, arXiv:0902.4022.

[4] J.M. Obrecht, R.J. Wild, M. Antezza, L.P. Pitaevskii, S. Stringari and E.A. Cornell, Phys. Rev. Lett. **98**, 063201 (2007).

[5] W.J. Kim, M. Brown-Hayes, D.A.R. Dalvit, J.H. Brownell and R. Onofrio, Phys. Rev. A **78**, 020101(R) (2008); A **79**, 026102 (2009); R.S. Decca et al., ibid. A **79**, 026101 (2009); S. de Man, K. Heeck and D. Iannuzzi, ibid. A **79**, 024102 (2009).

[6] I. Pirozhenko, A. Lambrecht and V.B. Svetovoy, New J. Phys. **8**, 238 (2006).

[7] M. Bostrom and B.E. Sernelius, Phys. Rev. Lett. **84**, 4757 (2000); B.E. Sernelius, *ibid.* **87**, 139102 (2001).

[8] B. Geyer, G.L. Klimchitskaya and V.M. Mostepanenko, J. Phys. A **40**, 13485 (2007).

[9] I. Brevik, S.A. Ellingsen and K.A. Milton, New J. Phys. **8**, 236 (2006).

[10] *Proccedings of the 8th Workshop on Quantum Field Theory under the Influence of External Conditions* (edited by M. Bordag and V.M. Mostepanenko) J. Phys. A **41**, (2008).

[11] M. Brown-Hayes *et al.*, J. Phys. A **39**, 6195 (2006); P. Antonini *et al.*, arXix:0812.5065.

[12] R.S. Decca, D. Lopez, E. Fischbach, G.L. Klimchitskaya, D.E. Krause and V.M. Mostepanenko, Ann. Phys. **318**, 37 (2005); Eur. Phys. J. C **51**, 963 (2007), Phys. Rev. D **75**, 077101 (2007).

[13] S. de Man, K. Heeck, R. J. Wijngaarden and D. Iannuzzi, arXiv:0901.3720.

[14] H. J., van Leeuwen, J. Phys. Radium **2**, 362 (1921); J. H. Van Vleck, *The Theory of Electric and Magnetic Susceptibilities* (Clarendon Press, Oxford, 1932).

[15] H. B. Callen and T. A. Welton, Phys. Rev **83**, 34 (1951); R. Kubo, Rep. Prog. Phys. **29**, 255 (1966); R. Kubo, in *Statistical Mechanics of Equilibrium and Non-Equilibrium Systems*, edited by J. Meixner (North-Holland, Amsterdam,1965), p. 81.

[16] H.B.G. Casimir, Rev. Mod. Phys. **17**, 343 (1945).

[17] G. Bimonte and E. Santamato, Phys. Rev. A **76**, 013810 (2007).

[18] G. S. Agarwal, Phys. Rev. A **11**, 230, 243, 253 (1975); A **12**, 1475, 1974 (1975).

[19] P. R. Buenzli and Ph. A. Martin, Phys. Rev. E **77**, 011114 (2008).

[20] V.B. Bezerra, G.L. Klimchitskaya and V.M. Mostepanenko, Phys. Rev. A **65**, 052113 (2002); **66**, 062112 (2002); V.B. Bezerra, G.L. Klimchitskaya, V.M. Mostepanenko and C. Romero, *ibid.* **69** 022119 (2004).

[21] M. Boström and B.E. Sernelius, Physica A **339**, 53 (2004).

[22] V.B. Svetovoy and M.V. Lokhanin, Phys. Rev. A **67**, 022113 (2003), V.B. Svetovoy and R. Esquivel, Phys. Rev. E **72**, 036113 (2005).

[23] S.A. Ellingsen, I. Brevik, J.S. Høye and K.A. Milton, quant-ph:0809.0763.

[24] G.Bimonte, E. Calloni, G. Esposito, L. Milano and L. Rosa, Phys. Rev. Lett. **94**, 180402 (2005); G. Bimonte, E. Calloni, G. Esposito, and L. Rosa, Nucl. Phys. B **726**, 441 (2005); G. Bimonte *et al.*, J. Phys. A **41**, 164023 (2008); G. Bimonte, Phys. Rev. A **78**, 062101 (2008).

[25] J. R. Roth, Nasa Tech. Note, Nasa TN D-3880 (1967).

[26] H.B.G. Casimir, Proc. K. Ned. Akad. Wet. Rev. **51**, 793 (1948).

[27] P. W. Milonni, Phys. Rev. A **25**, 1315 (1982).

[28] G. Bimonte, arXiv:0902.4537.

[29] L.P. Pitaevskii, Phys. Rev. Lett. **101**, 163202 (2008); D.A.R. Dalvit and S.K. Lamoreaux, ibid. **101**, 163203 (2008); V.B. Svetovoy, ibid. **101**, 163603 (2008).